J. Fluid Mech. (2001), *vol.* 434, *pp.* 101–118. Printed in the United Kingdom

Nonlinear stability of a fluid-loaded elastic plate with mean flow

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(Received 2 May 2000 and in revised form 23 October 2000)

It has been known for some time that the unsteady interaction between a simple elastic plate and a mean flow has a number of interesting features, which include, but are not limited to, the existence of negative-energy waves (NEWs) which are destabilized by the introduction of dashpot dissipation, and convective instabilities associated with the flow–surface interaction. In this paper we consider the nonlinear evolution of these two types of waves in uniform mean flow. It is shown that the NEW can become saturated at weakly nonlinear amplitude. For general parameter values this saturation can be achieved for wavenumber k corresponding to low-frequency oscillations, but in the realistic case in which the coefficient of the nonlinear tension term (in our normalization proportional to the square of the solid–fluid density ratio) is large, saturation is achieved for all k in the NEW range. In both cases the nonlinearities act so as increase the restorative stiffness in the plate, the oscillation frequency of the dashpots driving the NEW instability decreases, and the system approaches a state of static deflection (in agreement with the results of the numerical simulations of Lucey *et al.* 1997). With regard to the marginal convective instability, we show that the wave-train evolution is described by the defocusing form of the nonlinear Schrödinger (NLS) equation, suggesting (at least for wave trains with compact support) that in the long-time limit the marginal convective instability decays to zero. In contrast, expansion about a range of other points on the neutral curve yields the focusing form of the NLS equation, allowing the existence of isolated soliton solutions, whose amplitude is shown to be potentially significant for realistic parameter values. Moreover, when slow spanwise modulation is included, it turns out that even the marginal convective instability can exhibit solitary-wave behaviour for modulation directions lying outside broad wedges about the flow direction.

1. Introduction

The unsteady interaction between a solid structure and the surrounding fluid is of considerable practical concern across a wide range of technological and biological applications, as well as being of fundamental interest in its own right. An apparently simple model problem involves a flat elastic plate driven by harmonic excitation, with a mean fluid flow above it. Two aspects of this problem will be of interest here. First, Landahl (1962) and Benjamin (1960, 1963) considered modal wave energy, being the work done in building up the mode from rest at time $t = -\infty$. They showed that over a range of frequencies there exist neutral modes with negative wave energy, which we refer to as negative-energy waves (NEWs) (also termed class A waves by Benjamin). These waves are unusual in that application of damping to the system, such as by addition of dashpots fixed to the plate, necessarily lowers the total system energy and thereby destabilizes the NEWs. This idea has been explored further by Cairns (1979), who derived a simple expression for the wave energy. Second, the fluid-loaded plate with mean flow possesses convective instabilities associated with the flow–surface interaction. Brazier-Smith & Scott (1984) and Crighton & Oswell (1991) showed that with uniform mean flow the system is convectively unstable for all non-zero flow speeds below some critical value, and is absolutely unstable for larger flow speeds. With a more realistic steady boundary layer profile, this behaviour is again found, but additional instability modes associated with the boundary layer are also present – see Carpenter & Garrad (1985, 1986) for full details.

The work described above has been completed using strictly linear theory. However, it is important to understand the effects of nonlinearity, particularly with regard to the evolution, and possible amplitude saturation, of the linear instabilities. The nonlinear motion of fluid-loaded plates in the absence of mean flow has been considered by a number of authors. For instance, in the regime of light fluid loading, where the most significant nonlinearity corresponds to the additional plate tension due to bending, Abrahams (1987) has considered the resonance of a finite elastic baffle. Sorokin (2000) has recently extended this work to heavy fluid loading, in which case nonlinear terms in the fluid–plate coupling must also be included. D. G. Crighton (1997, personal communication) has suggested considering the nonlinear behaviour of infinite fluid-loaded plates with mean flow, with the inclusion of yet further nonlinearities to account for geometrical effects in the plate. This suggestion is now investigated in this paper, first in the context of a NEW destabilized by the presence of plate dashpots, then for the (marginal) convective instability mode mentioned in the previous paragraph and then finally for other neutral modes. We restrict attention here to strictly uniform mean flow. We also consider only infinite plates, although it should be noted that the behaviour of finite plates can in some respects be quite different – see for example Lucey (1998).

In § 2 we describe the governing equations. The NEWs and convective instabilities we are interested in are found only under heavy fluid loading conditions, and this means that the nonlinearities described by Sorokin (2000) must certainly be included. In § 3 we consider the evolution of a NEW excited by damping, and it is shown that the application of standard weakly nonlinear analysis to a mode of the form $\exp(\mathrm{i}kx - \mathrm{i}\omega t)$ at fixed ω does not lead to saturation of the linear instability, except in the special case when ω is very small, implying, rather unrealistically, that the NEW will typically grow to $O(1)$ amplitude. However, we are able to show that small-amplitude saturation can be achieved for all frequencies in the NEW range in an alternative way, in which the nonlinear tension term in the plate equation is allowed to feature at leading order (the coefficient of this term in our non-dimensional equations is in fact proportional to the square of the solid–fluid density ratio, which is clearly large in many practical applications). The mechanism for amplitude saturation in this case is shown to be that the growth of the linearly unstable NEW leads to an increase in the plate-induced tension, and hence to an increase in the effective plate stiffness opposing the destabilizing effects of the fluid. The growth rate of the instability is then reduced, leading to a reduction in the rate of working of the damping, and hence, since the rate of working is proportional to ω^2, to a decrease in the oscillation frequency. Equilibrium is reached when the plate stiffness and hydrodynamic loading balance, for which $\omega = 0$ and the plate is in a state of static deflection. A generalization of Cairn's (1979) expression for the wave energy is also presented.

In § 4 we consider weakly nonlinear analysis of the (marginal) convective instability,

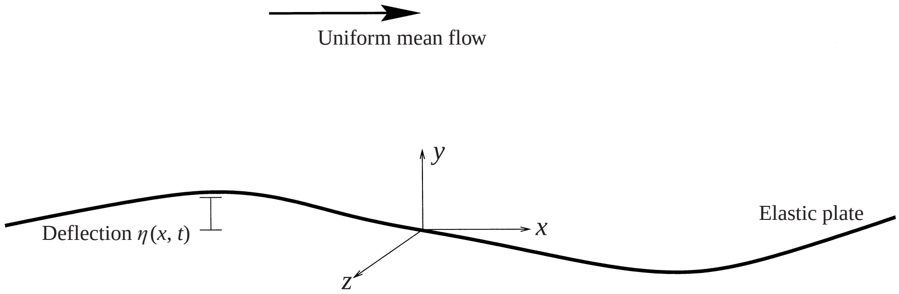

FIGURE 1. The geometry of the system.

and show that the spatio-temporal evolution of the wave train is governed by the nonlinear Schrödinger (NLS) equation with real coefficients. The signs of these coefficients predict wave defocusing, leading to the conclusion that the convective instability will typically decay in the long-time limit. However, we show in § 5 that the focusing form of the NLS equation is obtained when one expands about a range of other points on the lower branch of the neutral curve, allowing the existence of isolated solitons. Moreover, we show in § 6 that when slow spanwise modulation is included even the convective instability can exhibit solitary-wave behaviour, for modulation directions lying outside (rather broad) wedges about the flow direction.

2. Problem formulation

2.1. *Governing equations*

We consider an infinitely long, thin elastic plate of uniform thickness h_* (here the suffix $_*$ is used to denote dimensional variables), and bending stiffness $B_* = E_* h_*^3/12(1-\nu^2)$, where E_* is Young's modulus and ν is the Poisson ratio. The plate has mass per unit area m_*, where $m_* = h_* \rho_*^s$ and ρ_*^s is the solid density. We suppose that the undisturbed plate lies in the plane $y_* = 0$ aligned along the x_*-axis. In $y_* < 0$ there is a vacuum, but in $y_* > 0$ there is an incompressible fluid of uniform density ρ_*^f, which in its undisturbed state has a uniform flow of speed U_* in the positive x_*-direction. The plate is assumed to be infinite in the spanwise direction, and (apart from in the final section) we will restrict attention to purely two-dimensional motion. We follow Crighton & Oswell (1991) and non-dimensionalize lengths by m_*/ρ_*^f and time, t_*, by $m_*^{5/2}/(\rho_*^f)^2 B_*^{1/2}$, so that for instance in these new units the dimensionless flow speed is $U = U_* m_*^{3/2}/\rho_*^f B_*^{1/2}$. In what follows we will work exclusively in dimensionless variables. The system is shown in figure 1.

One important question to address straightaway is whether we are to consider the regimes of either light fluid loading or heavy fluid loading. Specifically, for a wave of dimensional frequency ω_* the light and heavy fluid loading regimes are usually taken to be defined by the ratio μ/k_p being small or large respectively, where $\mu = \rho_*^f/m_*$ and k_p is the plate bending wavenumber with $k_p^4 = m_* \omega_*^2/B_*$. This ratio is proportional to the ratio of the mass of fluid in one bending wavelength to the mass per unit area of the plate, and reduces to simply $\omega^{-1/2}$, where ω is the non-dimensional frequency. It will be seen in § 2.2 that the linear instabilities we wish to analyse occur at very low values of ω for realistic flow speeds, and it therefore follows that we must consider the limit of heavy fluid loading. The consequence of this, as pointed out by Sorokin

(2000), is that nonlinear effects must be included in both the plate equation and in the equations describing the boundary condition between the fluid and the plate.

Nonlinear effects must certainly be included in the plate equation for the flexural deflection $\eta(x,t)$. D. G. Crighton (1997, personal communication) has suggested that these will correspond to inclusion of the full nonlinear expression for the plate curvature in the elastic term and the calculation of the hydrodynamic pressure on the genuine plate surface $y = \eta(x,t)$ rather than on $y = 0$. Another nonlinear effect arises from the tension induced in the plate due to bending – see Dowell (1975), Abrahams (1987). Previous work on this extra tension term has tended to focus on *finite* elastic baffles. In order to incorporate the tension within a model of an infinite plate, we follow Sorokin (2000) and suppose that the plate has periodically arranged hinges at $x = ns$, $n = 0, \pm 1, \pm 2, \ldots$, which allow vertical, but not horizontal, deflection of each fixing point. When this tension term is included we are therefore forced to restrict attention to disturbances of wavelength $s \equiv 2\pi/k$, and higher harmonics, and the nonlinear tension in the plate can be determined by integrating over a single wavelength. In this paper we will first consider the effects of both classes of plate nonlinearity on the stability of NEWs. We will then neglect the tension term (this is equivalent to removing the hinges and supposing that the nonlinear tension is relaxed to zero along the infinite plate), and consider the effects of the remaining nonlinearities on convective instabilities.

Following the arguments advanced in the previous paragraph we arrive at the nonlinear plate equation

$$\frac{\partial^2}{\partial x^2}\left[\frac{\eta_{xx}}{(1+\eta_x^2)^{3/2}}\right] + \eta_{tt} - \frac{\tau}{s}\left(\int_0^s \eta_{x'}^2 \,\mathrm{d}x'\right)\eta_{xx} + \epsilon c_d \eta_t = -p(x, y = \eta, t). \tag{2.1}$$

Here the first term on the left corresponds to the elastic restoring force (i.e. the second derivative of plate curvature). A number of models of plate dynamics (e.g. Sorokin 2000) neglect the nonlinear effects of non-zero curvature, but this will be included here as it seems to be an appropriate geometric effect for an infinite plate. However, as will be noted in the final section, this nonlinearity has no qualitative effect on the results. The second term in (2.1) is the plate inertia, and the third term is the nonlinear tension due to bending, where $\tau = 6(1-\nu^2)(\rho_*^s/\rho_*^f)^2$ (see Abrahams 1987). Note that only the leading-order term, consistent with subsequent analysis, has been included in the induced tension. The fourth term on the left of (2.1) represents the effects of dampers, or dashpots, attached to the underside of the plate. These dashpots provide a drag force which is proportional to speed, with dimensionless drag coefficient ϵc_d (ϵ is included here for the purposes of subsequent asymptotic analysis, in which $\epsilon \to 0$ and $c_d = O(1)$). The inclusion of dashpots is a key element in this paper, because we will subsequently be studying the evolution of NEWs, which are destabilized by dissipation. Of course, the issue of the form of nonlinear plate equations is a difficult one, especially for finite systems, and the reader is referred to Paidoussis (1998) for a full discussion of these issues as they relate to slender structures.

On the right-hand side of (2.1), $p(x,y,t)$ is the hydrodynamic pressure which provides the fluid loading. The hydrodynamic pressure is given by the (nonlinear) Bernoulli equation

$$p = -\phi_t - U\phi_x - \tfrac{1}{2}(\phi_x^2 + \phi_y^2), \tag{2.2}$$

where the velocity potential $\phi(x,y,t)$ satisfies Laplace's equation

$$\nabla^2\phi = 0. \tag{2.3}$$

By restricting attention to the irrotational flow of an incompressible fluid, the fluid motion is described exactly by Laplace's equation without the need to include further nonlinearity. In addition, we must apply a boundary condition between the plate and the fluid, and this corresponds to the condition of the continuity of vertical velocity, which takes the form

$$\phi_y = \eta_t + (U + \phi_x)\eta_x \tag{2.4}$$

applied on the actual surface $y = \eta(x,t)$.

Equations (2.1)–(2.4) describe the nonlinear problem to be considered in this paper, and it is worth briefly comparing these equations with similar systems which have been studied previously. Abrahams (1987) considered a finite elastic baffle under light fluid loading, without damping. In this limit it is argued that the Bernoulli equation can be linearized, the hydrodynamic loading is evaluated on $y = 0$ and the only significant nonlinearity in the plate equation is the tension due to bending term. Sorokin (2000) has considered the equivalent problem to that of Abrahams, again without the damping terms, but in the limit of heavy fluid loading. In that model the same nonlinear plate equation as in Abrahams (1987) is used, but now the nonlinearity in equations (2.2) and (2.4) is included as described here. One difference between our present study and Sorokin (2000) is that the latter considers compressible flow, and argues that to the appropriate order of approximation the unsteady flow in the bulk of the fluid satisfies the linear wave equation. Finally, we note that an initial study of equations (2.1)–(2.4), but without the tension due to bending and damping terms, has been completed by D. G. Crighton (1997, personal communication).

2.2. *Linearized solution*

When all the nonlinear terms in equations (2.1)–(2.4) are neglected, the boundary condition (2.4) is applied on the mean surface $y = 0$ and the dashpot term is removed, we are left with precisely the linearized system studied by Brazier-Smith & Scott (1984) and Crighton & Oswell (1991). By considering harmonic disturbances proportional to $\exp(\mathrm{i}kx - \mathrm{i}\omega t)$, these authors are able to derive a dispersion relation of the form

$$D(k,\omega) \equiv \omega^2 - k^4 + \frac{(\omega - kU)^2}{|k|} = 0, \tag{2.5}$$

where $|k| = \pm k$ when the real part of k is positive and negative respectively. It turns out that the system is absolutely unstable for $U > U_c = 0.0742\ldots$, while for $U < U_c$ the dispersion diagram takes the form shown in figure 2.

There are two key features to be noted from the linear behaviour. First, on the lower branch of the neutral curve the modes are NEWs for $k_b < k < k_0$, where $k_0 = U^{2/3}$ is the point where $\omega = 0$. The idea of the wave energy was introduced by Landahl (1962) and Benjamin (1963), and corresponds to the work done in slowly building up the wave starting from rest at time $t = -\infty$. For this linearized system, Cairns (1979) shows that the wave energy, W_0, is given for positive k by

$$W_0 \equiv \mathscr{E}|A|^2 \equiv \frac{\omega}{4}\frac{\partial D}{\partial \omega}|A|^2 = \frac{\omega}{2}\left[\omega + \frac{(\omega - kU)}{k}\right]|A|^2, \tag{2.6}$$

and it is a straightforward matter to show that $\mathscr{E}$ is indeed negative on the lower branch for positive ω. Since the excitation of a NEW reduces the overall system energy, it follows that NEWs are destabilized, according to linear theory, by the introduction of dashpots. Second, for $0 < \omega < \omega_s(U)$ the system is spatially convectively unstable. Specifically, for each ω in this range there exists a conjugate pair of complex roots

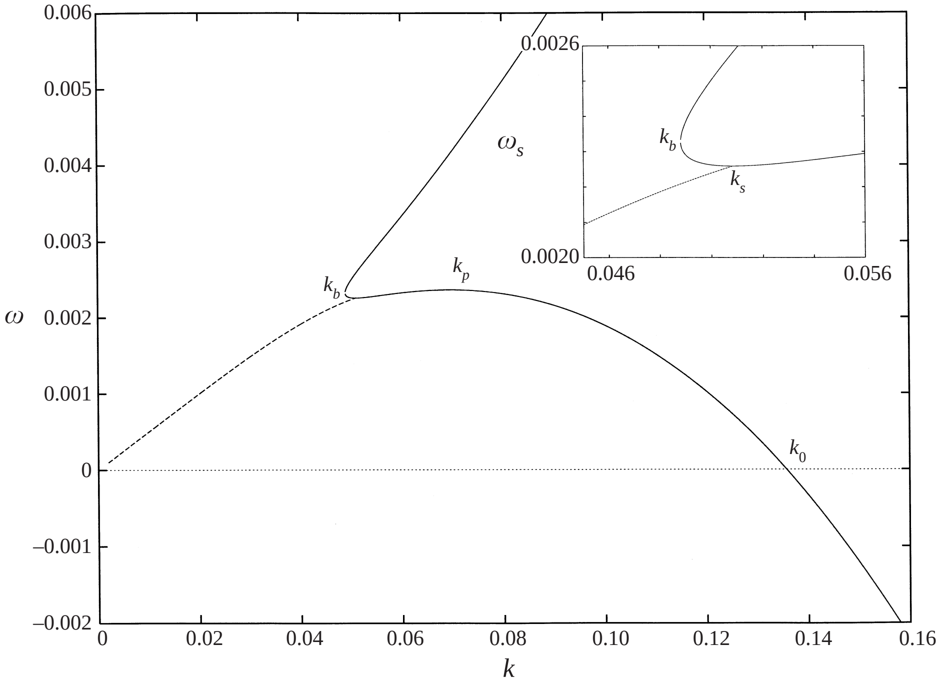

FIGURE 2. The linear dispersion diagram, describing equation (2.5). Here $U = 0.05$. A magnified view of the region around k_b is also shown. Solid lines represent neutral modes, and the dashed line is the real part of $k(\omega)$ for the convective instability.

in the k-plane, and it is shown in Brazier-Smith & Scott (1984) that the root with negative imaginary part, $k(\omega)$ say, is indeed a downstream-propagating convective instability. In the rest of this paper we will consider the nonlinear development of first the NEWs and then the (marginal) convective instabilities. Of course, once the damping is introduced the NEWs also correspond to convective instabilities, apart from at the turning point $k = k_p$, which becomes an absolute instability.

It should be noted from figure 2 that the various coordinates are small, and indeed Crighton & Oswell (1991) have shown that in the limit of small U we have $k_{s,b} = O(U)$, $\omega_{s,b}(U) = O(U^2)$, while the remaining turning point on the lower branch scales as $k_p = O(U^{2/3})$, $\omega_p = O(U^{5/3})$. This limit of small U is entirely realistic in practice (for 2 cm thick steel in water the case $U = 0.05$ plotted in figure 2 corresponds to a dimensional flow speed of 9.8 m s^{-1}, which is towards the top end of the range of feasible speeds underwater), and this therefore confirms the assertion made earlier that we must consider heavy fluid loading, since clearly $\omega \ll 1$ for the waves we are interested in.

3. Negative-energy waves

We have already noted that the linearized version of the problem possesses NEWs, which are destabilized by the introduction of dashpots. Specifically, if we consider a neutral mode on the lower branch with wavenumber in the range $k_b < k < k_0$ then $\mathscr{E}$ is negative, and if the (linearized) dashpot term is introduced, as in (2.1), then by introducing the slow time $T_1 = \epsilon t$ it is straightforward to show that the wave amplitude $A(T_1)$ satisfies

$$\frac{\mathrm{d}A}{\mathrm{d}T_1} + \frac{c_d\omega^2}{4\mathscr{E}}A = 0, \tag{3.1}$$

so that the wave grows, with amplitude proportional to $\exp(-\omega^2\epsilon c_d t/4\mathscr{E})$. In this section we aim to determine the nonlinear temporal evolution of this wave. The spatial wavenumber k is held fixed, in order to include the nonlinear tension term, as described in § 2.1. We consider three cases: the first with the tension coefficient τ taken to be $O(1)$ and ω arbitrary within the NEW range, second with $\tau = O(1)$ and ω small, and third with $\tau \gg 1$ and ω again arbitrary. In the first case it will be suggested that saturation can only occur at $O(1)$ amplitude, while in the second and third cases weakly nonlinear saturation will be achieved.

3.1. $\tau = O(1)$

We attempt weakly nonlinear analysis of the problem by writing

$$\left.\begin{aligned} \eta(x,t) &= \epsilon^{1/2}[AE + \text{c.c}] + \epsilon[\eta_1^{(2)}A^2E^2 + \eta_1^{(1)}AE + \text{c.c.} + \eta_1^{(0)}] + \cdots, \\ p(x,y,t) &= \epsilon^{1/2}(Ap_0^{(1)}E\exp(-ky) + \text{c.c.}) + \epsilon(p_1^{(2)}A^2E^2\exp(-2ky) \\ &\quad + p_1^{(1)}AE\exp(-ky) + p_1^{(0)}|A|^2 + \text{c.c.}) + \cdots, \end{aligned}\right\} \tag{3.2}$$

with a similar expression for $\phi(x,y,t)$, where $E = \exp(\mathrm{i}kx - \mathrm{i}\omega t)$. In the first instance we suppose that ω and τ are $O(1)$. At $O(\epsilon^{1/2})$ we find exactly the linear problem of Crighton & Oswell, with $D(k,\omega) = 0$ and the leading-order amplitudes of the pressure and velocity potential on $y = 0$ being $p_0^{(1)} = -(\omega - kU)^2/k$ and $\phi_0^{(1)} = \mathrm{i}(\omega - kU)/k$ respectively. At $O(\epsilon)$ we find that

$$\left.\begin{aligned} \eta_1^{(2)} &= -\frac{(\omega - kU)^2}{D(2k,2\omega)}, \\ p_1^{(2)} &= -\frac{(\omega - kU)^2}{D(2k,2\omega)}\left\{8\omega^2 - 32k^4 + \frac{2(\omega - kU)^2}{k}\right\}, \\ \phi_1^{(2)} &= \frac{-\mathrm{i}p_1^{(2)}}{2(\omega - kU)}, \\ p_1^{(0)} &= -2(\omega - kU)^2, \end{aligned}\right\} \tag{3.3}$$

with all other unknown quantities at $O(\epsilon)$ being zero (note that if $D(k,\omega) = 0$ then it is easy to show that $D(2k,2\omega)$ is strictly negative). After some algebra, we then find that at $O(\epsilon^{3/2})$ we have the amplitude equation

$$\frac{\mathrm{d}A}{\mathrm{d}T_1} + \frac{c_d\omega^2}{4\mathscr{E}}A + \mathrm{i}\beta_1 A|A|^2 = 0, \tag{3.4}$$

where

$$\beta_1 = \frac{4k(\omega - kU)^2 - 3k^6 - \dfrac{4(\omega - kU)^4}{D(2k,2\omega)} + 4\tau k^4}{4\left(\omega + \dfrac{(\omega - kU)}{k}\right)} \tag{3.5}$$

is real. The solution of this Landau equation is simply that $|A(T_1)|$ grows exponentially exactly as in the linear system (3.1), and only the phase of $A(T_1)$ undergoes nonlinear modification.

In order to attempt to find a nonlinear saturated state one is therefore led to attempt to balance the linear growth with a quintic term. Specifically, if one now takes $\eta = O(\epsilon^{1/4})$ and introduces two slow time scales $T_1 = \epsilon^{1/2}t$ and $T_2 = \epsilon t$ then after a considerable amount of algebra it turns out that

$$\frac{\partial A}{\partial T_1} + \mathrm{i}\beta_1 A|A|^2 = 0, \qquad \frac{\partial A}{\partial T_2} + \frac{c_d\omega^2}{4\mathscr{E}}A + \mathrm{i}\beta_2 A|A|^4 = 0, \tag{3.6}$$

where β_2 is also real. Hence, once again the nonlinear term acts only to modify the phase of $A(T_1, T_2)$ and does not saturate the linear instability. Indeed, it seems that one could continue this process indefinitely, choosing scalings to balance successively higher nonlinearities, but on each occasion the coefficient of the nonlinear term would be imaginary. This sort of behaviour suggests that the NEW could only be stabilized at $O(1)$ amplitude, as observed by Smith & Burgraff (1985) and Moston, Stewart & Cowley (2000) in the context of instability wave behaviour in the flat-plate boundary layer. The reason for this is the fact that the term driving the linear instability (i.e. $\epsilon c_d \eta_t$) is apparently exactly out of phase with the nonlinear restoring terms (typically of the form η_x^2 etc.)

It does not seem possible to complete an $O(1)$ amplitude analysis of the complicated equations (2.1)–(2.4), even if one were to omit the geometrical nonlinearities contained in the plate equation and restrict attention to just the tension due to bending and the nonlinearities in the boundary condition. However, it turns out that there are two alternative limits which will in fact lead to the more expected situation of saturation at small amplitude. We now consider these possibilities in turn.

3.2. $\tau = O(1)$, *low frequency*

There is one circumstance when the weakly nonlinear analysis described above will actually yield a saturated solution, and that corresponds to the limit of small ω. We write $\omega = \epsilon\bar{\omega}$ with $\bar{\omega} = O(1)$ and again use the scaling $\eta = O(\epsilon^{1/2})$ as in (3.2), with slow times $T_1 = \epsilon t$ and $T_2 = \epsilon^2 t$. At $O(\epsilon^{3/2})$, equation (3.4) becomes simply

$$\frac{\partial A}{\partial T_1} + \mathrm{i}\beta_1 A|A|^2 = 0, \tag{3.7}$$

where in the limit of small ω we have $k \to k_0 = U^{2/3}$ and

$$\beta_1 \to -\tfrac{9}{28}U^3 - \tau U^{5/3}. \tag{3.8}$$

At $O(\epsilon^{5/2})$ we then find the equation

$$\frac{\partial A}{\partial T_2} + \mathrm{i}\beta_3 A|A|^4 - \frac{\mathrm{i}}{2U}\left[c_d\frac{\partial A}{\partial T_1} - \mathrm{i}\bar{\omega}c_d A\right] = 0, \tag{3.9}$$

where β_3 is real. Using (3.7) it follows that, since $\beta_1 < 0$, this equation does indeed result in a saturated state, with

$$|A| \to \left[\frac{\bar{\omega}}{\frac{9}{28}U^3 + \tau U^{5/3}}\right]^{1/2} \tag{3.10}$$

as $T_2 \to \infty$. It should be noted that although this saturated amplitude is formally $O(\epsilon^{1/2})$, its coefficient could be large, given that U is often small in practice.

The mechanism which has provided the amplitude saturation for small ω is simply the fact that the nonlinearities act to reduce (since β_1 is negative) the oscillation frequency of the dashpot term $\epsilon c_d \partial\eta/\partial t$ over the first slow time-scale T_1 according to equation (3.7). Over the second slow time scale T_2 the growth rate of the NEW is proportional to the oscillation frequency of the dashpot motion, and since the rate of working of the dashpot is proportional to the square of its velocity, saturation is achieved when this frequency is forced to zero. Of course this relies on ω being small, and the same result is certainly not achieved in the previous subsection, where the term driving the instability of the NEW, $-\mathrm{i}\omega\epsilon c_d A$, is necessarily asymptotically larger than the nonlinear phase adjustment term $\epsilon^2 c_d \partial A/\partial T_1$. We note from (3.8) that β_1 contains contributions from both the tension due to bending term (proportional to τ) and the other nonlinearities associated with the boundary conditions and the plate geometry (with net effect proportional to U^3 in (3.8)). These terms are both negative, so that both sorts of nonlinearity have the effect of slowing down the oscillation. In this limit it follows that the presence of the tension due to bending term (with the associated constraints described in § 2.1) is not necessary to achieve amplitude saturation.

3.3. *Large* τ

The factor τ multiplying the tension term in (2.1) is in fact a large number in many practical situations, owing to the fact that it is proportional to the squared solid-to-fluid density ratio $(\rho_*^s/\rho_*^f)^2$ – for steel in water we have $\nu = 0.3$ and $\rho_*^s/\rho_*^f = 7.8$, so that $\tau = 332.2$. This suggests that the nonlinear tension term could in fact play a more significant role in the plate dynamics than is implied by the above analysis, in which we implicitly assumed that $\tau = O(1)$. We will therefore now consider the realistic case of τ large, and take the preferred limit $\tau = \epsilon^{-2}\bar{\tau}$ with $\bar{\tau} = O(1)$. We suppose that ω takes any value in the NEW range. The appropriate scaling for the plate deflection now turns out to be

$$\eta(x,t) = \epsilon[AE + \text{c.c.}] + \cdots, \tag{3.11}$$

with $A = A(T)$ and $T = \epsilon t$, but this time we must allow the frequency of the motion to vary, so that now

$$E = \exp\left(\mathrm{i}kx - \frac{\mathrm{i}}{\epsilon}\int^{T}\omega(T')\,\mathrm{d}T'\right). \tag{3.12}$$

Substituting this expansion, and equivalent expansions for the potential and pressure, into (2.1)–(2.4) we find at $O(\epsilon)$ the amplitude-dependent dispersion relation

$$D^{\mathscr{N}\mathscr{L}}(k,\omega,|A|) \equiv \omega^2 - k^4 + \frac{(\omega - Uk)^2}{|k|} - 2k^4\bar{\tau}|A|^2 = 0. \tag{3.13}$$

This can be compared with the linearized dispersion relation (2.5), and it becomes clear that interestingly the effect of the nonlinear tension is not to introduce a conventional tension (which would appear as a term proportional to k^2 in the dispersion relation), but rather is to increase the effective stiffness of the plate (i.e. the amplitude term appears as the coefficient of k^4). At $O(\epsilon^2)$ we then find the solvability condition

$$\frac{\mathrm{d}|A|}{\mathrm{d}T}\left\{2\omega + \frac{2(\omega - kU)}{k}\right\} + |A|\frac{\mathrm{d}\omega}{\mathrm{d}T}\left(1 + \frac{1}{k}\right) + \omega c_d|A| = 0, \tag{3.14}$$

and, using (3.13) to determine $\mathrm{d}\omega/\mathrm{d}T$, the amplitude equation follows as

$$\frac{\mathrm{d}|A|}{\mathrm{d}T} + \frac{c_d\omega^2}{4\mathscr{E}_{NL}}|A| = 0, \tag{3.15}$$

where

$$\mathscr{E}_{NL} = \mathscr{E} + \frac{(k+1)\omega^2\bar{\tau}k^3|A|^2}{4\mathscr{E}}. \tag{3.16}$$

Equation (3.15) is solved with ω given as a function of $|A|$ from (3.13); specifically

$$\omega = \frac{U \pm [k^4 + k^3 - U^2k + 2k^3(k+1)\bar{\tau}|A|^2]^{1/2}}{1 + (1/k)}. \tag{3.17}$$

If we were to linearize these equations in amplitude, then the dispersion relation (3.13) would reduce to the linear dispersion relation (2.5), $\mathscr{E}_{NL}$ would reduce to $\mathscr{E}$, the frequency ω would become constant and equation (3.15) would reduce to equation (3.1).

We now recall that the wave energy is defined to be the work done in slowly building up the wave to its current amplitude, starting from rest at time $t = -\infty$, and it is shown in the Appendix that the wave energy in our system is

$$W(T) = \int_{-\infty}^{T} \mathscr{E}_{NL} \frac{\mathrm{d}|A|^2}{\mathrm{d}T'} \mathrm{d}T'. \tag{3.18}$$

This is the nonlinear generalization of the wave energy W_0 identified by Cairns (1979) for the linear problem (given here in equation (2.6)). It is clear that, in contrast to W_0, $W(T)$ is not only time-dependent but also depends on the precise way in which the wave amplitude is built up from zero. From the derivation of (3.15) it is clear that the differences between $W(T)$ in (3.18) and W_0 in (2.6) have arisen from the fact that the frequency ω now evolves on the slow time scale T, which is in turn due to the fact that the amplitude now appears in the dispersion relation for ω at fixed k. In the linear theory of Cairns, the sign of W_0 is precisely the same as the sign of $\mathscr{E}$, and in our nonlinear theory a similar result holds; provided that we suppose that the amplitude of the wave is built up from rest monotonically, i.e. $\mathrm{d}|A|^2/\mathrm{d}T > 0$, it follows that the sign of $W(T)$ is precisely the same as the sign of $\mathscr{E}_{NL}$. Following on from (3.18), equation (3.15) can now be interpreted as being the equality between the rate of change of the wave energy and the rate of working of the dashpots averaged over a period of the fast oscillation.

Waves with initially positive linearized wave energy W_0 are damped by the addition of dissipation according to linear theory. From (3.16), $\mathscr{E}_{NL} > \mathscr{E} > 0$ for these modes, so that from (3.15) it follows that $|A|$ still decays in T. Moreover, such a mode corresponds to the positive root in (3.17), so that ω decreases with increasing T. As $T \to \infty$ we therefore have $|A| \to 0$, and ω approaches the value predicted by linear theory for the given value of k. In contrast, for waves with initially negative linearized wave energy, it follows that $\mathscr{E}_{NL} < \mathscr{E} < 0$, so that with the addition of dissipation the wave amplitude grows with T. These modes correspond to the negative root in (3.17), so that as $|A|$ increases ω decreases, but this time ω decreases towards zero as $T \to \infty$. This results in the saturated state in which $\omega \to 0$ and, from (3.13),

$$|A| \to |A|_s = \left[\frac{kU^2 - k^4}{2k^4\bar{\tau}}\right]^{1/2}. \tag{3.19}$$

It is easy to see that $|A|_s$ is a real number, given that k has been chosen so that the wave has $\mathscr{E} < 0$ initially. It also follows that $\mathscr{E}_{NL} \to 0$ as $T \to \infty$, but clearly the wave energy itself will not approach zero – the wave energy in the saturated state will of course correspond to the total energy extracted from the system by the dashpots over $-\infty < T < \infty$. Sample plots of the evolution of A for various values of wavenumber

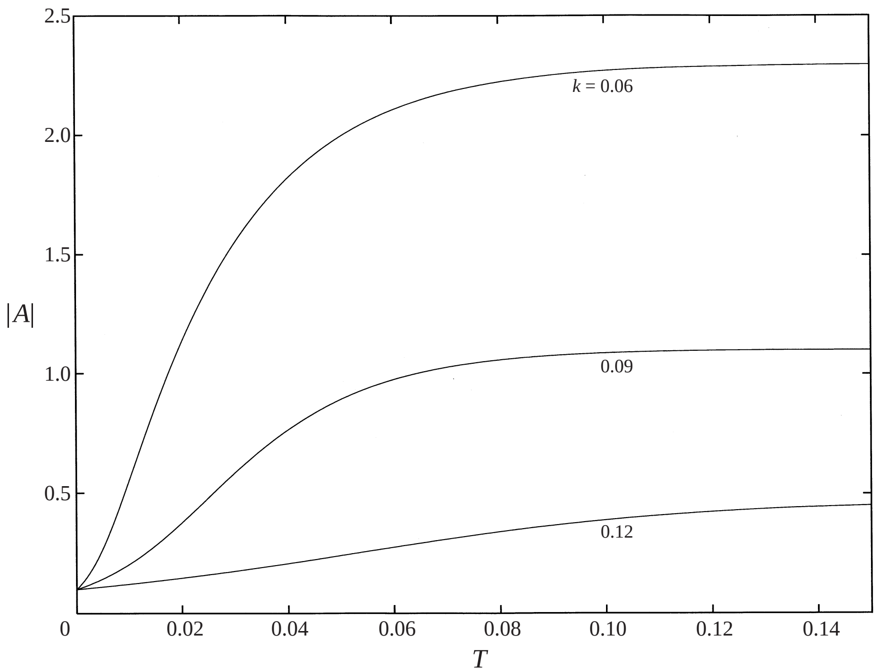

FIGURE 3. Evolution of wave amplitude $|A|$ with slow time T, for $U = 0.05$, $|A(0)| = 0.1$, $\bar{\tau} = 1$ and various values of k in the NEW range.

k for which $\mathscr{E}_{NL} < 0$ are shown in figure 3; these can be obtained in a very simple manner by numerical integration of (3.15) using standard routines. Of course, for initial amplitudes less than $|A|_s$ the wave amplitude grows as described above, as seen in figure 3. For initial amplitudes in excess of $|A|_s$, it is easy to show that the mode on the minus branch of (3.17) has $\omega < 0$ initially, so that $\mathscr{E}$, and hence $\mathscr{E}_{NL}$, are positive; the wave then decays, ω approaches zero from below and $|A|$ approaches the saturation value $|A|_s$ from above.

As already noted, the saturated amplitude is of size $O(\epsilon)$, and is therefore formally small. However, the coefficient $|A|_s$ of this amplitude given by (3.19) will not itself be small for $O(1)$ values of $\bar{\tau}$; typically k and U are small, and it is shown by Crighton & Oswell (1991) that the NEWs are described by either the scaling $k = O(U^{2/3})$ (around the points $k = k_{p,0}$ in figure 2), in which case $|A|_s$ is an $O(1)$ number, or by $k = O(U^2)$ (around the points $k = k_{b,s}$ in figure 2), in which case $|A|_s$ is $O(U^{-1/2})$, and is therefore large. This behaviour is confirmed in figure 3; the lowest value of k shown there is close to $k = k_b$ in linear theory, leading to the highest saturated amplitude.

The instability can be thought to arise from the way in which the destabilizing force due to the fluid loading exceeds the restoring stiffness in the plate. It follows that the weakly nonlinear mechanism for the saturation of NEWs is that the growth in the wave amplitude excited by the dashpots increases the induced tension of the plate, which in turn increases this restorative stiffness, thereby reducing the growth rate. Equilibrium is then reached when the fluid and plate forces are balanced. Since the rate of working of the dashpots is proportional to ω^2, it also follows that the reduction in growth rate leads to a reduction in frequency, until saturation occurs at $\omega = 0$, at which point the plate deflection is static and the dashpots do no more work. Of course, this is equivalent to the effect described in the previous subsection, in which the (small) frequency experiences a nonlinear correction, which in effect reduces the oscillation frequency of the dashpot to zero. It should be emphasized, however, that the action of the nonlinear tension term with large τ will result in saturation for all ω within the NEW range. It is pleasing to note that our prediction of a saturated state of static deflection is in agreement with the results of numerical simulations performed by Lucey *et al.* (1997). The non-zero static deflection of a finite elastic baffle has been studied by Ellen (1997).

Finally we note that when the mean flow has a boundary-layer profile it has been shown originally by Miles (1957) for water waves, and then in the present context by Benjamin (1959, 1963), that the critical layer in the flow provides an out-of-phase pressure component which acts to stabilize a NEW (see also § 3 of Carpenter & Garrad 1986). In the present problem of uniform mean flow there is no critical layer, of course, so that the mechanism for stabilizing the NEWs described in this section does not rely on the steady-profile curvature.

4. Convective instability

We now move on to study the nonlinear spatial and temporal evolution of the (marginal) convectively unstable waves. We will have in mind the downstream spatial and temporal evolution of a wave train excited by a driver oscillating at a given temporal frequency, which seems to be a scenario of relevance to fluid–structure interaction problems. This will in effect mean that we will complete the weakly nonlinear analysis about the neutral mode $k = k_s$, $\omega = \omega_s$, and we will now use the small parameter ϵ to denote the size of the amplitude of the wave packet. Of course, this analysis will not apply all along the convectively unstable branch, where the growth rates can become quite significant. In order to obtain a cubic nonlinearity at the appropriate order, it can easily be seen that we must expand the plate displacement as

$$\eta = \epsilon(AE + \text{c.c.}) + \epsilon^2(\eta_1^{(2)}A^2E^2 + \eta_1^{(1)}AE + \eta_1^{(0)}|A|^2 + \text{c.c.}) + \cdots, \tag{4.1}$$

where $E = \exp(\mathrm{i}k_s x - \mathrm{i}\omega_s t)$ and A is again the amplitude of the leading-order term in the plate deflection. From figure 2 we have that $D_k(k_s, \omega_s) = 0$ (this follows from the identity $\omega_k = -D_k/D_\omega$ and the fact that D_ω remains finite for $k \neq 0$), suggesting introducing the slow spatial and temporal scales $X = \epsilon x$, $Y = \epsilon y$ and $T = \epsilon^2 t$, with $A = A(X, T)$. Expansions for the pressure and potential, similar to (4.1) but involving additional dependence on y and Y, can be written down, and together with (4.1) are substituted into the equations described in § 2.1. It proves convenient, however, to write the pressure on $y = 0$ in the form

$$\epsilon(Ap_0^{(1)}E + \text{c.c.}) + \epsilon^2(p_1^{(2)}A^2E^2 + p_1^{(1)}AE + p_1^{(0)}|A|^2 + \text{c.c.}) + \cdots, \tag{4.2}$$

with a similar expression for the potential on the surface.

Because we wish to consider both temporal and spatial evolution, we will here neglect the nonlinear tension due to bending term in (2.1) by setting $\tau = 0$. This is because the introduction of this tension term requires us to consider fixed wavelength disturbances, prohibiting the spatial modulation we want to include here. In physical terms, removal of the nonlinear tension corresponds to adopting the scenario in which the induced tension is relaxed to zero along the infinite plate. The dashpot term is also removed, since the convective instability can be driven purely by the plate inertia and stiffness and the mean flow.

At $O(\epsilon)$ we find exactly the linear problem of Crighton & Oswell, as in § 3.1. At

$O(\epsilon^2)$ we find that

$$\left.\begin{aligned} &\eta_1^{(2)} = -\frac{(\omega_s - Uk_s)^2}{D(2k_s, 2\omega_s)}, \quad p_1^{(2)} = 2\mathrm{i}(\omega_s - Uk_s)\phi_1^{(2)}, \\ &\phi_1^{(2)} = \frac{\mathrm{i}(\omega_s - Uk_s)}{k_s}\eta_1^{(2)} + \mathrm{i}(\omega_s - Uk_s), \\ &p_1^{(1)} = -\frac{\mathrm{i}}{k_s}\frac{\partial A}{\partial X}(\omega_s^2 - k_s^2 U^2), \quad \phi_1^{(1)} = -\frac{\omega_s}{k_s^2}\frac{\partial A}{\partial X}, \\ &p_1^{(0)} = -(\omega_s - k_s U)^2, \end{aligned}\right\} \tag{4.3}$$

with all other unknown quantities at $O(\epsilon^2)$ being zero. At this order we have used the fact that $D_k(k_s, \omega_s) = 0$.

At $O(\epsilon^3)$ we can now find the evolution equation for $A(X, T)$ by considering terms proportional to E. After a great deal of algebra we find the equation

$$\frac{D_{kk}}{2}\frac{\partial^2 A}{\partial X^2} - \mathrm{i}D_\omega \frac{\partial A}{\partial T} = lA|A|^2, \tag{4.4}$$

where the derivatives of $D(k, \omega)$ are evaluated at (k_s, ω_s) and the coefficient of the cubic nonlinearity is given by

$$l = \frac{3k_s^6}{2} - \frac{2(\omega_s - Uk_s)^4}{D(2k_s, 2\omega_s)} + 2k(\omega_s - k_s U)^2. \tag{4.5}$$

Using $D(k, \omega) = 0$, it is easy to show that $D(2k, 2\omega) < 0$ for any neutral mode, and therefore that $l > 0$ always. In addition, it is straightforward to show that $D_\omega(k_s, \omega_s) < 0$ (this is in fact equivalent to the statement made in § 2.2 that the modes on the lower branch are NEWs), and that $D_{kk}(k_s, \omega_s) > 0$ (given that $D_\omega(k_s, \omega_s) < 0$, this follows from the fact that the turning point on the neutral curve at $k = k_s$ is a minimum). In the limit of small U, Crighton & Oswell show that $k_s = O(U)$, $\omega_s = O(U^2)$ and $\omega_s - k_s U = O(U^{5/2})$, so that $D_\omega = O(U^{3/2})$, $D_{kk} = O(U)$ and the three terms in l are all of the same, relatively small, size $O(U^6)$. Equation (4.4) is the nonlinear Schrödinger (NLS) equation with real coefficients (see Craik 1985; Peregrine 1985; Johnson 1997), and the relative signs of the coefficients mean that this is the so-called NLS minus form, which describes wave defocusing. For compact initial data, as one might expect in some practical fluid-loading problems (say the generation of a finite wave packet by a source oscillating over a finite time interval), it is believed that in most cases the amplitude $A(X, T)$ will decay in time like $T^{-1/2}$ (see Craik 1985, p. 200). This specifically answers the question we posed at the beginning as to the nonlinear evolution of the marginal convective instability – in the long-time limit the wave train will eventually defocus and will decay to zero amplitude. Given the relatively small size of the cubic coefficient l, however, one might expect that the wave train would grow to relatively large amplitude linearly before the nonlinear effects become important.

5. Other neutral waves

It is of course possible to complete the weakly nonlinear analysis of the previous section about any point on the neutral curve, and, subject to making a Galilean transformation so as to eliminate a term of the form $\mathrm{i}D_k \partial A/\partial X$, one would again end up with exactly the NLS equation (4.4), but now with the derivatives of D and the various terms in l evaluated at the new neutral point. The numerical values of these coefficients will of course change: specifically, in the region around $k = k_p$ it follows from § 2.2 that for small U we have $D_\omega = O(U)$, $D_{kk} = O(U^{4/3})$ and all terms in l are of the same size, $O(U^4)$.

We already know that $D_\omega < 0$ all along the lower branch of the neutral curve. Since the turning point $k = k_p$ is a maximum, it is easy to see that there is a point on the lower branch, $k = k_i$ say with $k_s < k_i < k_p$, where D_{kk} vanishes, such that $D_{kk} > 0$ for $k < k_i$ and $D_{kk} < 0$ for $k > k_i$ (for $U = 0.05$, $k_i = 0.062\ldots$). The significance of this is that when $k > k_i$ the NLS equation (4.4) then takes the plus, or focusing, form (see Peregrine 1985), allowing soliton solutions with $|A| \to 0$ as $|X| \to \infty$ to exist (Zakharov & Shabat 1972). Specifically, the one-soliton solution of (4.4) is (see Craik 1985, p. 200)

$$A(X,T) = a\left|\frac{D_{kk}}{l}\right|^{1/2} \operatorname{sech}\left[a\left(X + \frac{bD_{kk}}{D_\omega}T\right)\right] \exp\left[\mathrm{i}bX - \frac{\mathrm{i}D_{kk}}{2D_\omega}(a^2 - b^2)T\right], \tag{5.1}$$

where a and b are arbitrary. It is particularly noteworthy that the envelope amplitude differs from the characteristic width by a factor $|D_{kk}/l|^{1/2}$, and from the remarks in the previous paragraph it follows that for small U this factor is $O(U^{-4/3})$, i.e. the soliton has a large amplitude relative to its width. It is therefore entirely reasonable to expect that the evolution of the wave train will be characterized by the appearance of a series of envelope solitons, the precise number of which being determined by the initial conditions. Moreover, these solitons will have an amplitude of size $O(\epsilon U^{-4/3})$, which must be strictly small within the confines of the weakly nonlinear analysis used here, but for practical values of U could well be significant, and therefore potentially observable.

6. Slow spanwise modulation

So far we have restricted attention to two-dimensional motion. However, if we were to consider three-dimensional disturbances, proportional to $\exp(-\mathrm{i}\omega t + \mathrm{i}kx + \mathrm{i}mz)$, with z the spanwise coordinate, then it is easy to show that the linearized dispersion relationship (2.5) becomes

$$D(k,m,\omega) \equiv \omega^2 - k^4 - m^4 - 2k^2m^2 + \frac{(\omega - kU)^2}{(k^2+m^2)^{1/2}} = 0; \tag{6.1}$$

essentially, this comes from replacing the fourth derivative in the linearized plate equation by ∇^4 and extending the dimension in the Laplace equation. The nonlinear generalization of the fluid equations (2.2)–(2.4) to account for $m \neq 0$ is straightforward, but the generalization of the plate equation (2.1) seems to be exceedingly difficult and will not be discussed further. However, progress can be made by supposing that although $m = 0$ the waves are slowly modulated in the spanwise direction by introducing the slow coordinate $Z = \epsilon z$ and setting $A = A(X,Z,T)$. In this way the effects of spanwise curvature are excluded from the plate equation to the order considered, with the effect that repeating the weakly nonlinear analysis of § 4, i.e. expanding about the point $k = k_s, \omega = \omega_s, m = 0$, yields the NLS equation

$$\frac{D_{kk}}{2}\frac{\partial^2 A}{\partial X^2} + \frac{D_{mm}}{2}\frac{\partial^2 A}{\partial Z^2} - \mathrm{i}D_\omega\frac{\partial A}{\partial T} = lA|A|^2, \tag{6.2}$$

where the coefficients l, D_{kk} and D_ω take the same value as in the strictly two-dimensional case. Using (6.1) we have

$$D_{mm} = -4k_s^2 - \frac{(\omega_s - k_s U)^2}{k_s^3}, \tag{6.3}$$

while D_{km} vanishes for $m = 0$. It therefore follows that D_{kk} and D_{mm} have opposite signs, leading to the possibility of the NLS equation taking either its plus or minus form in different angular sectors in the (X, Z)-plane, in parallel to what is found by Hui & Hamilton (1979) for the Kelvin ship-wave problem. Specifically, if we consider modulation only in the direction $\bar{X}$ inclined at an angle ϕ to the X-direction, then it follows that inside the wedges $\tan\phi = \tan\phi_s$, where $|\tan\phi_s| = |D_{kk}/D_{mm}|^{1/2}$, we have the minus (defocusing) form of the NLS equation, while outside the wedges we have the plus (focusing) form of the NLS equation. It is easy to see that this wedge angle is large, and specifically $\tan\phi_s = O(U^{-1/2})$ for small U. This therefore suggests that once spanwise modulation is included, the marginal convective instability can indeed evolve into isolated solitary waves, but only along modulation directions lying outside wide wedges about the flow direction.

Of course, the spanwise modulation can be included for any neutral mode, as in § 5. From (6.3) we see that D_{mm} is negative for all neutral modes, while we have already noted that D_{kk} approaches zero from above as one moves along the neutral curve from k_s to k_i. From this it follows that the wedge angle ϕ_s narrows to zero as k_i is approached, allowing solitary solutions over an increasingly wide range of directions in (X, Z)-space. For $k > k_i$, the NLS equation will take its plus (focusing) form in all modulation directions.

7. Concluding remarks

In this paper we have investigated the nonlinear evolution of two distinct types of instability waves in the canonical problem of a fluid-loaded elastic plate with mean flow. We have demonstrated that a NEW destabilized by damping can be stabilized without the need to introduce mean-flow shear, either for $O(1)$ values of the tension coefficient τ and low oscillation frequency, or for large τ and any frequency within the NEW range. The nonlinearities act to increase the plate restoring stiffness until it balances the destabilizing effects of the fluid. Since the rate of working of the dashpots is proportional to the square of the oscillation frequency, it follows that the oscillation frequency is constantly reduced, and in the saturated state we have $\omega = 0$ and the amplitude is then constant. Divergence instability is an important concept in aero-elasticity, and for the infinite plate considered here corresponds to the excitation of the NEWs by even a very small amount of damping. In the present problem divergence instability can then be identified on the lower branch of the dispersion curve in figure 2 between the points (k_b, ω_b) and $(k_0, 0)$. In this language, our conclusion is that the amplitude of the divergence instability will grow in time, but that the nonlinear effects will act to decrease its frequency down to zero, so that it evolves into a static deflection of constant amplitude. This is in agreement with the numerical simulations of Lucey *et al.* (1997).

We have also analysed the nonlinear behaviour of the convective instability. For strictly two-dimensional motion we predict that the wave train will be defocused and will decay in time, but once slow spanwise modulation is included it turns out that, owing to the second derivatives of the dispersion function being of opposite sign, wave focusing, and hence isolated soliton solutions, can occur in modulation directions lying outside rather broad wedges about the flow direction. Wave focusing can also be seen when expanding about a range of other neutral modes, even for strictly two-dimensional motion. The amplitude of possible isolated soliton solutions is shown to be potentially significant in practice.

We have already noted, at the end of § 3, that it has been known for some time that steady-flow shear can also stabilize a NEW. Inclusion of this effect within a nonlinear scheme, presumably using the stabilizing component of the hydrodynamic pressure to push the linear growth rate to higher order where it is balanced by the cubic nonlinearities from the plate equation, may be possible, and this is being investigated further. Other aspects to be considered include the evolution of absolute instability in these problems, and the effects of finite, as opposed to infinite, plate length. The precise development of particularly steep nonlinear waves along the wedge directions $\tan\phi = \tan\phi_s$, exactly as done by Hui & Hamilton (1979) for the ship-wave problem, could also be of some interest. We also note that NLS solitons have also been found by Watanbe & Sugimoto (2000), in the context of wave motion along an elastic beam.

Finally, we note that the results of this paper remain qualitatively unchanged if the nonlinear curvature term is linearized in our plate equation (i.e. if the first term in (2.1) is replaced by η_{xxxx}). This change would mean that the elastic term in our plate equation would be identical to that used by Abrahams (1987) and Sorokin (2000). Moreover, it would then match the standard von Kármán theory of a plate (see for example Stoker 1968). Quantitatively, removal of this effect would mean the following: deletion of the term $3k^6$ in the numerator of (3.5), so that the saturated amplitude for the NEW with low frequency is modified, with the term $9U^3/28$ in the denominator of (3.10) replaced by $15U^3/14$; and for the nonlinear stability of the neutral modes, the term $3k_s^6/2$ in the coefficient l is removed (note that this has no effect on the sign of l, and hence on the existence of solitary waves or otherwise). Qualitatively, this modification of the plate equation therefore has no effect on the conclusions reached in this paper.

This work has benefited from many fruitful discussions with Professor David Crighton FRS (1942–2000), to whose memory this paper is dedicated with gratitude and affection. Financial support provided by the US Office of Naval Research under grant N00014-96-1-1085 is acknowledged. Conversations with Professor S. Sorokin are also gratefully acknowledged.

Appendix

In this Appendix we present a physical derivation of the nonlinear wave energy W. The analysis is in exact parallel with that of Cairns (1979), who investigated linear problems with fixed frequency. The idea is to consider a wave whose amplitude is built up slowly from rest starting at time $t = -\infty$. We therefore write, in line with the multiple scale approach used in § 3, the plate deflection as

$$\eta(x,t) = A(t)\exp\left(\mathrm{i}kx - \mathrm{i}\int_{-\infty}^{t}\omega(t')\mathrm{d}t'\right), \tag{A 1}$$

where the amplitude A and frequency ω vary only slowly with time t, while the wavenumber k remains fixed. At each instant in time, ω, k and $|A|$ are related by the dispersion relation (3.13). In order to allow the wave (A 1) to build up from rest, an external force needs to be applied to match the difference between the force exerted on the plate by the fluid and the force exerted on the fluid by the plate. As the plate vibrates this external force does work (either positive or negative), thereby providing the wave energy. The pressure exerted by the fluid on the plate is determined simply from the linearized Bernoulli equation and Laplace's equation, while the pressure exerted by the plate on the fluid is

$$\eta_{xxxx} + \eta_{tt} - \frac{\tau}{s}\left(\int_0^s \eta_{x'}^2 \,\mathrm{d}x'\right)\eta_{xx}\,; \tag{A 2}$$

recall that the only nonlinearity which is relevant to the NEW described in § 3.3 is the tension due to bending, so that the other nonlinear terms in the plate equation and boundary conditions (2.1)–(2.4) are ignored here. Using (A 1), together with similar expressions for the fluid pressure and potential, it follows that the external pressure which needs to be imposed is

$$-\mathrm{i}D_\omega^{\mathscr{N}\mathscr{L}}\frac{\mathrm{d}A}{\mathrm{d}t} - \mathrm{i}\frac{D_{\omega\omega}^{\mathscr{N}\mathscr{L}}}{2!}A\frac{\mathrm{d}\omega}{\mathrm{d}t}. \tag{A 3}$$

The plate normal velocity is $-\mathrm{i}\omega A$ (neglecting the slow evolution of A), and it is therefore an easy matter to show that the rate of working of the external pressure, averaged over a period of the fast oscillation, is

$$\frac{\omega D_\omega^{\mathscr{N}\mathscr{L}}}{4}\frac{\mathrm{d}|A|^2}{\mathrm{d}t} + \frac{|A|^2}{4}\frac{D_{\omega\omega}^{\mathscr{N}\mathscr{L}}}{2!}\frac{\mathrm{d}\omega^2}{\mathrm{d}t}. \tag{A 4}$$

This expression can now be simplified by using (3.13) to convert the time derivative of ω into the time derivative of $|A|$, and then integrating from $-\infty$ up to t yields the wave energy $W(t)$ in the form

$$W(t) = \int_{-\infty}^{t} \mathscr{E}_{NL}\frac{\mathrm{d}|A|^2}{\mathrm{d}t}\mathrm{d}t, \tag{A 5}$$

where

$$\mathscr{E}_{NL} = \frac{\omega D_\omega^{\mathscr{N}\mathscr{L}}}{4} - \frac{D_{\omega\omega}^{\mathscr{N}\mathscr{L}} D_{|A|}^{\mathscr{N}\mathscr{L}}\omega|A|}{8D_\omega^{\mathscr{N}\mathscr{L}}}. \tag{A 6}$$

This expression for the wave energy simplifies exactly to the result derived by Cairns when ω is independent of time and the amplitude-dependent term is removed from the dispersion relation.

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